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Optimal Rational Starting Approximations

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1. INTRODUCTION

In [7] D. G. Moursund examined the problem of approximating $x^{1/2}$ over an interval $[a, b]$ $(a > 0)$ by applying the Newton–Raphson iteration scheme to classes of polynomial and rational approximants. G. Meinardus and G. D. Taylor [6] have observed that the above problem may be posed in a more general setting and the results obtained by Moursund extended to the approximation of additional functions. Specifically, given a compact subset X of $[a, b]$, let $C(X)$ denote the space of all continuous real-valued functions defined on X normed by $\|f\| = \max\{|\omega(x) f(x)|: x \in X\}$, where $\omega \in C(X)$ and $\omega > 0$ on X. Let K be a convex subset of $C(X)$ and Φ a continuous mapping of K into $C(X)$. The problem of interest was then to approximate $g \in \Phi(K)$ by elements of $\Phi(M)$ where M is a subset of K consisting of members of a Haar subspace of $C[a, b]$. By imposing certain restrictions on Φ, K and M, Meinardus and Taylor were able to develop a theory for the above nonlinear approximation problem which is analogous to the classical Chebyshev theory. Moreover they were able to obtain results similar to those of P. H. Sterbenz and C. T. Fike [13] and R. F. King and D. L. Phillips [2] in the more general setting. Since the theory obtained has application to many iterative processes that can be used to approximate functions such as e^x, $\ln x$ and $x^{1/n}$, $n = 2, 3,\ldots$ the following definition is made:

DEFINITION. $p \in M$ is an optimal (or best) starting approximation for g, with respect to Φ and M, if $\| g - \Phi(p)\| \leqslant \| g - \Phi(q) \|$ for all $q \in M$.

In this paper we shall investigate the above problem for M a subset of K consisting of certain classes of rational functions. We shall be particularly concerned with the characterization of optimal starting approximations for certain choices of K. Uniqueness of best approximation follows from the characterization theorems as in the classical rational Chebyshev approximation. Existence in the general setting is difficult to establish. The theory obtained will be applied to the Newton operator and optimal starting approximations calculated for x^α, $\alpha \in (0, 1)$, e^x, and $\ln x$. Finally we shall indicate a number of additional iteration schemes for which the theory is also applicable.

2. Definitions and Basic Results

In what follows X will denote a compact subset of $[a, b]$ containing at least $m + n + 2$ points where m and n are fixed for the discussion at hand. Let P_n denote the set of all polynomials p with $\partial p \leqslant n$ (∂p denotes the degree of p) and set

$$R_{m,n}[a, b] = \{p \mid q: p \in P_n, q \in P_m, (p, q) = 1, q > 0 \text{ on } [a, b]\},$$

where $(p, q) = 1$ denotes the fact that the polynomials p and q are relatively prime.

In our development we wish to make use of the results and techniques of the classical Chebyshev rational approximation. Consequently we introduce the following definitions and results from the work of Meinardus and Taylor [6]. We assume that K, M and Φ are as described in Introduction.

Definition 2.1. The operator Φ is called pointwise strictly monotone at $f \in K$ if for each $h, k \in K$ we have

$$|\Phi(h)(x_0) - \Phi(f)(x_0)| < |\Phi(k)(x_0) - \Phi(f)(x_0)| \qquad \text{for each } x_0 \in X$$

where either $k(x_0) < h(x_0) \leqslant f(x_0)$ or $f(x_0) \leqslant h(x_0) < k(x_0)$.

Lemma 2.2. *Let $\Phi: K \to C(X)$ be pointwise strictly monotone at $f \in K$. If $k \in K$ and at $x_0 \in X$, $k(x_0) \neq f(x_0)$, then $\Phi(k)(x_0) \neq \Phi(f)(x_0)$.*

Definition 2.3. The operator Φ is said to be pointwise fixed at $f \in K$ if $h \in K$ with $h(x_0) = f(x_0)$ for $x_0 \in X$ implies $\Phi(h)(x_0) = \Phi(f)(x_0)$.

Next we shall note that the composition of two continuous operators possessing the above properties is again such an operator provided that domains and ranges mesh correctly.

Lemma 2.4. *Let $\Phi: K \to C(X)$ and $\psi: L \to C(X)$ be continuous operators with $\Phi(K) \subseteq L$, Φ pointwise strictly monotone and pointwise fixed at $f \in K$ and ψ pointwise strictly monotone and pointwise fixed at $\Phi(f) \in L$. Then $\psi\Phi: K \to C(X)$ is a continuous pointwise strictly monotone operator at f which is also pointwise fixed at f.*

In general the notions of pointwise strictly monotone and pointwise fixed are independent. However if $\Phi: K \to K$, as is necessary if we wish to iterate with Φ, then the notions of pointwise strictly monotone and pointwise fixed are related according to the following lemma:

LEMMA 2.5. *If $\Phi: K \to K$ is a continuous operator on the convex set K which is pointwise strictly monotone at $f \in K$ then Φ is pointwise fixed at f.*

For the sake of any iteration processes we may be interested in, we state the following corollary to the above lemma:

COROLLARY 2.6. *If $\Phi: K \to K$ is continuous, $\Phi(f) = f$ for some $f \in K$ and Φ is pointwise strictly monotone at $f \in K$ then $\Phi^m: K \to K$ defined inductively by $\Phi^m(h) = \Phi(\Phi^{m-1}(h))$, $m = 2, 3, \ldots$, satisfies $\Phi^m(f) = f$ and Φ^m is pointwise strictly monotone at f.*

3. CHARACTERIZATION OF OPTIMAL STARTING APPROXIMATIONS

We shall now specialize M and K and develop an alternation theory for characterizing optimal starting approximations which is similar to the classical Chebyshev theory. The following version of a well-known lemma from Rice [11, p. 79] will be used extensively.

LEMMA 3.1. *Given $r = p/q \in R_{m,n}[a, b]$, $\tau > 0$ and any ordered set $\{x_i\}$ of s points in X, $s < 1 + \max\{m + \partial p, n + \partial q\}$, there is a rational function $r_\epsilon \in R_{m,n}[a, b]$ such that (i) $\| r_\epsilon - r \|_\infty < \tau$ and (ii) $\operatorname{sgn}(r(x) - r_\epsilon(x)) = (-1)^{i+1}$, $x \in (x_i, x_{i+1})$, $i = 0, 1, \ldots, s$, where $x_0 = a$ if $x_1 \neq a$ and $x_{s+1} = b$ if $x_s \neq b$.*

THEOREM 3.2. *Let $\Phi: K \to C(X)$ be a continuous operator, where K is a convex subset of $C(X)$. Let $M = K \cap R_{m,n}[a, b]$ be a nonempty, relatively open subset of $R_{m,n}[a, b]$. Finally, assume that Φ is pointwise strictly monotone and pointwise fixed at $f \in K \sim M$. Then $r \in M$ is the best starting approximation for $\Phi(f)$ if and only if there exist points*

$$\{x_i\}_{i=1}^N \subseteq X, \qquad N = 2 + \max\{m + \partial p, n + \partial q\},$$

for which

(i) $x_1 < x_2 < \cdots < x_N$,

(ii) $| \omega(x_i)(\Phi(f)(x_i) - \Phi(r)(x_i))| = \| \Phi(f) - \Phi(r)\|$, $i = 1,\ldots, N$, *and*

(iii) $\operatorname{sgn}(f(x_i) - r(x_i)) = (-1)^{i+1} \operatorname{sgn}(f(x_1) - r(x_1))$, $i = 1,\ldots, N$.

Proof. (*Sufficiency*). Since $f \notin M$ we know that there exists a point $x_0 \in X$ for which $f(x_0) \neq r(x_0)$. Thus $\| \Phi(f) - \Phi(r)\| \neq 0$. Suppose that $| \omega(x_i)(\Phi(f)(x_i) - \Phi(r)(x_i))| = \| \Phi(f) - \Phi(r)\|$ and

$$\operatorname{sgn}(f(x_i) - r(x_i)) = (-1)^{i+1} \operatorname{sgn}(f(x_1) - r(x_1)), \qquad i = 1,..., N.$$

Then $\Phi(f)(x_i) \neq \Phi(r)(x_i)$. Let $r_1 \in M$ be such that $\| \Phi(f) - \Phi(r_1)\| \leqslant \| \Phi(f) - \Phi(r)\|$. At x_i, $i = 1,..., N$, $| \omega(x_i)(\Phi(f)(x_i) - \Phi(r_1)(x_i))| \leqslant | \omega(x_i)(\Phi(f)(x_i) - \Phi(r)(x_i))|$. Now at each x_i either $r(x_i) > f(x_i)$ or $r(x_i) < f(x_i)$ since Φ is pointwise fixed at f. In the first case we have $r(x_i) \geqslant r_1(x_i)$ and in the second case that $r(x_i) \leqslant r_1(x_i)$ by the pointwise strict monotonicity of Φ at f. But this implies $r \equiv r_1$ as in the standard rational theory.

(*Necessity*). Suppose that there exists $\{x_i\}_{i=1}^{N'} \subseteq X$. $N' < N$, N' maximal on which $| \omega(x_i)(\Phi(f)(x_i) - \Phi(r)(x_i))| = \| \Phi(f) - \Phi(r)\|$ and

$$\operatorname{sgn}(f(x_i) - r(x_i)) = (-1)^{i+1} \operatorname{sgn}(f(x_1) - r(x_1)).$$

Let $I_1, I_2, ..., I_{N'}$, be a collection of relatively open intervals in $[a, b]$ such that $x_i \in I_i$, $\bar{I}_i \cap \bar{I}_j = \varphi$ for $i \neq j$ ($\bar{I}_i$ denotes the closure of I_i relative to $[a, b]$), all extreme points $= \{x \in X: | \omega(x)(\Phi(f)(x) - \Phi(r)(x))| = \| \Phi(f) - \Phi(r)\|\} \subseteq \bigcup_{i=1}^{N'} I_i$ and for each extreme point in I_i the function $f - r$ has constant sign. Let

$$Y = X \cap \left(\bigcap_{i=1}^{N'} I_i' \right),$$

where I_i' denotes the complement of I_i with respect to $[a, b]$. Y is a compact subset of X and $| \omega(x)(\Phi(f)(x) - \Phi(r)(x))| < \| \Phi(f) - \Phi(r)\|$ for all $x \in Y$. By continuity there exists $\rho > 0$ for which

$$\max_{x \in Y} | \omega(x)(\Phi(f)(x) - \Phi(r)(x))| \leqslant \| \Phi(f) - \Phi(r)\| - \rho.$$

Next, let

$$W_i = \{x \in X \cap \bar{I}_i : | \omega(x)(\Phi(f)(x) - \Phi(r)(x))| \geqslant \tfrac{1}{2}\| \Phi(f) - \Phi(r)\|$$

and

$$\operatorname{sgn}(f(x) - r(x)) = \operatorname{sgn}(f(x_i) - r(x_i))\}.$$

$W = \bigcup_{i=1}^{N'} W_i$ is a compact subset of X and so by continuity there exists $\eta > 0$ such that $|f(x) - r(x)| \geqslant \eta$ on W. Set

$$Z_i = \{x \in X \cap \bar{I}_i : | \omega(x)(\Phi(f)(x) - \Phi(r)(x))| \geqslant \tfrac{1}{2}\| \Phi(f) - \Phi(r)\|$$

and

$$\operatorname{sgn}(f(x) - r(x)) \neq \operatorname{sgn}(f(x_i) - r(x_i))\}$$

and let $Z = \bigcup_{i=1}^{N'} Z_i$. Observe that

$$|\,\omega(x)(\Phi(f)(x) - \Phi(r)(x))| < \|\,\Phi(f) - \Phi(r)\| \qquad \text{for all } x \in Z$$

by the construction of the intervals $\{\bar{I}_i\}$. Finally, set

$$U_i = \{x \in X \cap \bar{I}_i : |\,\omega(x)(\Phi(f)(x) - \Phi(r)(x))| \leqslant \tfrac{1}{2}\|\,\Phi(f) - \Phi(r)\|\}$$

and let $U = \bigcup_{i=1}^{N'} U_i$. Then by continuity there exists $\delta > 0$, $\delta \leqslant \rho$, such that

$$\max_{x \in Z \cup U} |\,\omega(x)(\Phi(f)(x) - \Phi(r)(x))| \leqslant \|\,\Phi(f) - \Phi(r)\| - \delta.$$

Using Lemma 3.1, the continuity of Φ and the fact that M is relatively open in $R_{m,n}[a, b]$ we can select $\epsilon_1 > 0$ such that for $0 < \epsilon \leqslant \epsilon_1$, $r_\epsilon \in M$ and

$$\max_{x \in Y \cup Z \cup U} |\,\omega(x)(\Phi(f)(x) - \Phi(r_\epsilon)(x))| \leqslant \|\,\Phi(f) - \Phi(r)\| - \tfrac{1}{2}\delta.$$

Next, by continuity of f and r, we can select ϵ_2, $0 < \epsilon_2 \leqslant \epsilon_1$, such that for $0 < \epsilon \leqslant \epsilon_2$, r_ϵ lies strictly between $f(x)$ and $r(x)$ on W. By the strict monotonicity of Φ at f we then have that

$$\max_{x \in W} |\,\omega(x)(\Phi(f)(x) - \Phi(r_\epsilon)(x))| < \|\,\Phi(f) - \Phi(r)\|.$$

Thus for ϵ with $0 < \epsilon \leqslant \epsilon_2$, r_ϵ is such that $\|\,\Phi(f) - \Phi(r_\epsilon)\| < \|\,\Phi(f) - \Phi(r)\|$. Finally, since M is relatively open in $R_{m,n}[a, b]$, we can select ϵ_3 with $0 < \epsilon_3 \leqslant \epsilon_2$ so that $r_{\epsilon_3} \in M$ and $\|\,\Phi(f) - \Phi(r_{\epsilon_3})\| < \|\,\Phi(f) - \Phi(r)\|$. Thus r_{ϵ_3} is a better starting approximation than r which is a contradiction and this concludes the proof of the theorem.

COROLLARY 3.3. *If $\Phi(f)$ has a best starting approximation under the setting of the above theorem then it is unique.*

In the next theorem we wish to consider a characterization of the best starting approximation for $\Phi(f)$ from a family of functions having restricted ranges. Much of the general theory for restricted range approximation by both polynomial and rational functions can be found in Taylor [14] and [15], Schumaker and Taylor [12], and Loeb, Moursund, and Taylor [5].

For our consideration let $l(x)$, $u(x) \in C(X)$ with $l(x) < u(x)$ for all $x \in X$. Define $K = \{f \in C(X): l(x) \leqslant f(x) \leqslant u(x)\}$ and set $M = K \cap R_{m,n}[a, b]$ which we assume is nonempty.

THEOREM 3.4. *Let $\Phi: K \to C(X)$ be a continuous operator which is pointwise strictly monotone and pointwise fixed at $f \in K \sim M$, where K and M are as defined above. Then $r = p/q \in M$ is a best starting approximation for $\Phi(f)$ if and only if there exists $\{x_i\}_{i=1}^N \subseteq X$, $N = 2 + \max\{n + \partial q, m + \partial p\}$, for which*

(i) $x_1 < x_2 < \cdots < x_N$,

(ii) $|\omega(x_i)(\Phi(f)(x_i) - \Phi(r)(x_i))| = \|\Phi(f) - \Phi(r)\|$, $r(x_i) = u(x_i)$, *or* $r(x_i) = l(x_i)$,

(iii) $\operatorname{sgn}^*(f(x_i) - r(x_i)) = (-1)^{i+1} \operatorname{sgn}^*(f(x_1) - r(x_1))$,

where

$$\operatorname{sgn}^*(f(x) - r(x)) = \begin{cases} \operatorname{sgn}(f(x) - r(x)) & \text{if } r(x) \neq l(x) \text{ and } r(x) \neq u(x) \\ +1 & \text{if } r(x) = l(x), \\ -1 & \text{if } r(x) = u(x). \end{cases}$$

COROLLARY 3.5. *Under the conditions of the above theorem the optimal starting approximation for $\Phi(f)$ is unique.*

In [9] A. Perrie examined the problem of rational approximation with osculatory interpolation. We shall now examine a problem analogous to that considered by Perrie for our operator setting. Let $X \subseteq [a, b]$ be compact, $\{y_i\}_{i=1}^p$ a fixed set of p points in X with $y_1 < y_2 < \cdots < y_p$ and $\{m_i\}_{i=1}^p$ a fixed set of positive integers with $m^* = \sum_{i=1}^p m_i < n + 1$. We shall assume that X contains at least $n + m - m^* + p + 2$ points. Furthermore, let $\{a_i\}_{i=1}^p$ be a fixed set of p real numbers and define

$$K = \{f \in C(X): f(y_i) = a_i, i = 1, 2, ..., p\}.$$

Let a_{ij}, $i = 1, ..., p$, $j = 0, 1, ..., m_i - 1$ be a second set of real numbers where $a_{i0} = a_i$, $i = 1, ..., p$ and set

$$M = \{r \in R_{m,n}[a, b]: r^{(j)}(y_i) = a_{ij}, i = 1, ..., p, j = 0, 1, ..., m_i - 1\}.$$

As before we are interested in approximating $g \in \Phi(K)$ by elements of $\Phi(M)$, where $\Phi: K \to C(X)$ is continuous.

In the case of ordinary rational approximation we know that for each $f \in C[a, b]$ there exists a best approximation from $R_{m,n}[a, b]$. However, in the case of interpolating rational functions we can no longer insure existence as is demonstrated by adapting the example of H. L. Loeb [3] to the operator setting.

For a fixed $r = p/q \in R_{m,n}[a, b]$ we shall write $P_n + rP_m$ to denote the subspace $\{p + rq: p \in P_n, q \in P_m\}$ of $C[a, b]$. Define

$$S(r) = \{h \in P_n + rP_m : h^{(j)}(y_i) = 0, i = 1,..., p, j = 0, 1,..., m_i - 1\}.$$

It is straightforward to verify that $S(r)$ is a $d = (1 + \max\{n + \partial q, m + \partial p\} - m^*)$-dimensional subspace of $C[a, b]$.

THEOREM 3.6. *Let $\Phi: K \to C(X)$ be a continuous operator which is pointwise strictly monotone and pointwise fixed at $f \in K \sim M$ where K and M have been defined above. Then the following are equivalent:*

(i) *$r^* = p^* \mid q^* \in M$ is a best starting approximation for $\Phi(f)$ from M.*

(ii) *The zero element $(0, 0,..., 0)$ is in the convex hull of the set of d-tuples $\{\sigma(x)\hat{x}: x \in X$ and $|\omega(x)(\Phi(f)(x) - \Phi(r^*)(x))| = \|\Phi(f) - \Phi(r^*)\|\}$, where $\sigma(x) = \operatorname{sgn}(f(x) - r^*(x))$ and $\hat{x} = (g_1(x),..., g_d(x))$ with $\{g_1,..., g_d\}$ a basis for $S(r^*)$. We shall write*

$$X(r^*) = \{x \in X: |\omega(x)(\Phi(f)(x) - \Phi(r^*)(x))| = \|\Phi(f) - \Phi(r^*)\|\}.$$

(iii) *There exist $d + 1$ consecutive points $x_1 < x_2 < \cdots < x_{d+1}$ in $X \sim \bigcup_{i=1}^{p} \{y_i\}$ such that*

(a) $|\omega(x_i)(\Phi(f)(x_i) - \Phi(r^*)(x_i))| = \|\Phi(f) - \Phi(r^*)\|,$
$i = 1, 2,..., d + 1,$

(b) $\operatorname{sgn}\{(f(x_i) - r^*(x_i))\Delta_i\} = (-1)^{i+1} \operatorname{sgn}\{(f(x_1) - r^*(x_1))\Delta_1\},$ $i = 1,..., d + 1$, *where*

$$\Delta_i = \begin{vmatrix} g_1(x_1) & g_1(x_2) \cdots g_1(x_{i-1}) & g_1(x_{i+1}) \cdots g_1(x_{d+1}) \\ \vdots & \vdots \quad\quad \vdots & \vdots \quad\quad \vdots \\ g_d(x_1) & g_d(x_2) \cdots g_d(x_{i-1}) & g_d(x_{i+1}) \cdots g_d(x_{d+1}) \end{vmatrix}.$$

(iv) *There exist $d + 1$ consecutive points $x_i \in X \sim \bigcup_{i=1}^{p} \{y_i\}$ such that*

(a) $|\omega(x_i)(\Phi(f)(x_i) - \Phi(r^*)(x_i))| = \|\Phi(f) - \Phi(r^*)\|, i = 1,..., d + 1,$

(b) $\operatorname{sgn}\{[f(x_i) - r^*(x_i)]\pi(x_i)\} = (-1)^{i+1} \operatorname{sgn}\{[f(x_1) - r^*(x_1)]\pi(x_1)\}$ *for* $i = 1,..., d + 1$, *where* $\pi(t) = (y_1 - t)^{m_1} \cdots (y_p - t)^{m_p}$ *if* $p \neq 0$ *and* $\pi(t) = 1$ *if* $p = 0$.

The above theorem is established by suitably modifying the arguments found in [4, Theorem 3.1, p. 286] for the operator setting.

THEOREM 3.7. *If $r^* \in M$ is an optimal starting approximation for $\Phi(f)$, then r^* is unique.*

4. Computation of an Optimal Starting Approximation

In this section we shall examine the problem of computation of an optimal starting approximation for an operator Φ. The characterization theorems we have developed are of particular importance in establishing computational procedures. From the classical theory an optimal starting approximation would be computed using a modified Remes algorithm which involves solving a nonlinear system of equations with Newton's method of higher order. However, if the operator Φ satisfies certain conditions then a best starting approximation can be more easily obtained. Moreover, the best starting approximation may be independent of the number of applications of Φ provided this iteration is well-defined. We now wish to consider sufficient conditions on Φ for which this behavior occurs. The following definitions and results are due to Meinardus and Taylor [6] and proofs will be omitted.

Definition 4.1. Let $\Phi: K \to C(X)$ be a continuous operator. We say that Φ possesses Property I at $f \in K$ provided for each $r \in K$ and $x, y \in X$, $r(x)/f(x) = r(y)/f(y)$ implies

$$\frac{\Phi(r)(x)}{f(x)} = \frac{\Phi(r)(y)}{f(y)}, \quad \text{and} \quad \frac{r(y)}{f(y)} < \frac{r(x)}{f(x)} \leqslant 1 \quad \text{or} \quad \frac{r(y)}{f(y)} > \frac{r(x)}{f(x)} \geqslant 1$$

implies

$$\left| 1 - \frac{\Phi(r)(x)}{f(x)} \right| < \left| 1 - \frac{\Phi(r)(y)}{f(y)} \right|.$$

Definition 4.2. Let $\Phi: K \to C(X)$ be a continuous operator. Φ is said to be one-sided at f provided either $\Phi(k) \geqslant \Phi(f)$ for all $k \in K$ or $\Phi(k) \leqslant \Phi(f)$ for all $k \in K$.

Theorem 4.3. *Let K be a convex subset of $C(X)$ and $\Phi: K \to C(X)$ be pointwise strictly monotone, pointwise fixed and possess Property I at $f \in K$ where $\Phi(f) = f$ and $f > 0$ on X. Norm $C(X)$ by $\| h \| = \| h/f \|_\infty (h \in C(X))$. Let $r = p \mid q \in R_{m,n}[a, b]$ (a, b fixed) be the best relative approximation to f from $R_{m,n}[a, b]$ with deviation λ; that is, $\| (f - r)/f \|_\infty = \inf\{\| (f - s)/f \|_\infty : s \in R_{m,n}[a, b]\}$. If $M = K \cap R_{m,n}[a, b]$ is nonempty and relatively open in $R_{m,n}[a, b]$ and $\delta r \in M$ for $\delta \in [1/(1 + \lambda), 1/(1 - \lambda)]$ then there exists $\delta_0 \in (1/(1 + \lambda), 1/(1 - \lambda))$ for which $\delta_0 r$ is the best starting approximation for f (with respect to Φ).*

Definition 4.1 and Theorem 4.3 have rather natural analogs in the setting of uniform approximation ($\omega(x) \equiv 1$).

DEFINITION 4.4. The operator Φ is said to possess Property J at $f \in K$ if for each $r \in K$ and $x, y \in X$, $f(x) - r(x) = f(y) - r(y)$ implies $\Phi(f)(x) - \Phi(r)(x) = \Phi(f)(y) - \Phi(r)(y)$, and $0 \leqslant f(x) - r(x) < f(y) - r(y)$ or $0 \geqslant f(x) - r(x) > f(y) - r(y)$ implies $|\Phi(f)(y) - \Phi(r)(y)| > |\Phi(f)(x) - \Phi(r)(x)|$.

THEOREM 4.5. *Let Φ: $K \to C(X)$, K a convex subset of $C(X)$, and $M = K \cap R_{m,n}[a, b]$ be a relatively open subset of $R_{m,n}[a, b]$ with $m \leqslant n$. Assume Φ is pointwise strictly monotone, pointwise fixed and possesses Property J at $f \in K \sim M$. If $r \in R_{m,n}[a, b]$ is the best uniform approximation to f from $R_{m,n}[a, b]$ with deviation λ and if $r + c \in M$ for $c \in [-\lambda, \lambda]$ then there exists $c_0 \in (-\lambda, \lambda)$ for which $r + c_0$ is the best starting approximation to $\Phi(f)$ from M.*

For the sake of any iteration we state the following theorem analogous to a result of Meinardus and Taylor [6].

THEOREM 4.6. *Let Φ: $K \to K$ satisfy the following properties:*

(i) *Φ is continuous*

(ii) *$\Phi(f) = f$*

(iii) *Φ is pointwise strictly monotone at $f \in K \sim M$*

(iv) *Φ is a one-sided operator at f*

(v) *Φ possesses Property I (Property J) at f.*

Then $\Phi^m = \Phi(\Phi^{m-1})$, $m = 2, 3,...$ has all the same properties as Φ and moreover the best starting approximation for $\Phi(f)$ is also the best starting approximation for $\Phi^m(f)$, $m = 2, 3,...$.

5. APPLICATION TO THE NEWTON OPERATOR

In this section we shall apply the theory developed to the operator associated with the well-known Newton iteration scheme. Set

$$S = \{f \in C^2(0, \infty): f', f'' \neq 0 \text{ for all } x > 0 \text{ and image } f = (0, \infty)\}.$$

The choice of $(0, \infty)$ here is quite arbitrary. For a fixed $x \in [a, b] \subseteq (0, \infty)$ we can solve the equation $f^{-1}(y) - x = 0$ by Newton's method to obtain the value of f at x where $f \in S$. In particular, if $y_0(x) = y(x)$ is the initial guess to $f(x)$ then the sequence defined inductively by

$$y_n(x) = y_{n-1}(x) - \{f^{-1}(y_{n-1}(x)) - x\}\{f'[f^{-1}(y_{n-1}(x))]\}, \qquad n = 1, 2,...$$

represents the Newton iteration scheme for determining the unique zero of the equation $f^{-1}(y) - x = 0$.

Our goal is to be able to calculate $f(x)$ for all $x \in [a, b]$ on a computer. To accomplish this we shall select a class of functions $M \subseteq R_{m,n}[a, b]$ which are easily programmed and then select a member of M as the initial guess. Precisely, we wish to find an element r of M for which

$$\left\| \frac{y_{n,r}(x) - f(x)}{f(x)} \right\|_\infty \leqslant \left\| \frac{y_{n,s}(x) - f(x)}{f(x)} \right\|_\infty$$

for all $s \in M$ where $y_{n,s}(x)$ denotes the nth Newton iterate at x with initial guess $s(x)$. This particular problem was first examined by D. G. Moursund and G. D. Taylor [8] and is a generalization of the subroutine used to calculate $x^{1/2}$ on a computer such as the CDC-3600.

In order to apply the theory we have developed we must first determine a suitable convex subset K of $C[a, b]$ such that the Newton operator N_f defined by

$$N_f(h)(x) = h(x) - \{f^{-1}(h(x)) - x\}\{f'[f^{-1}(h(x))]\}$$

maps K to K (so we may iterate). In [6] the following is established:

Case (i). If $f \in S$ and either $f' > 0$ and $f'' < 0$ on $(0, \infty)$ or $f' < 0$ and $f'' < 0$ on $(0, \infty)$ then

$$K = \{h \in C[a, b]: h(x) > 0 \quad \text{for all } x \in [a, b]\}$$

is a suitable choice for K.

Case (ii). If $f \in S$ and either $f' > 0$ and $f'' > 0$ or $f' < 0$ and $f'' > 0$ on $(0, \infty)$ then there exists a function $\varnothing_f \in C[a, b]$ such that if

$$K = \{h \in C[a, b]: 0 < h(x) < \varnothing_f(x) \qquad \text{for all } x \in [a, b]\}$$

then $N_f : K \to K$.

It is relatively straightforward to show that N_f is continuous, $N_f(f) = f$ and N_f is pointwise strictly monotone at f. Moreover, if f is in Case (i) then N_f is one-sided from above at f and for f in Case (ii) N_f is one-sided from below at f.

As previously mentioned the existence of a best starting approximation is in general a rather difficult problem. For the special choice of $f(x) = x^\alpha$, $\alpha \in (0, 1)$ or $f(x) = e^x$ existence has been established and, moreover, the best starting approximation is independent of the number of iterations. For $f(x) = x^\alpha$, $\alpha \in (0, 1)$, set

$$K = \{h \in C[a, b]: h(x) > 0 \text{ for all } x \in [a, b]\},$$

where $0 < a < b$. The operator N_f is then defined by

$$N_f(h)(x) = \alpha \left[\left(\frac{1}{\alpha} - 1 \right) h(x) + \frac{x}{h^{(1/\alpha)-1}(x)} \right]$$

for $h \in K$. In the case of $f(x) = e^x$, set

$$K = \{h \in C[a, b]: 0 < h(x) < e^{1+x} \text{ for all } x \in [a, b]\}$$

and define

$$N_f(h)(x) = h(x)(1 + x - \ln h(x))$$

for $h \in K$.

THEOREM 5.1. *For $m = 1, 2,...$ the following is true*:

(a) *The best starting approximation for m Newton iterations for the calculation of x^α is $\gamma_\alpha r_\alpha$ where r_α denotes the best relative approximation to x^α from $R_{m,n}[a, b]$ with deviation λ_α and*

$$\gamma_\alpha = \left[\frac{(1 + \lambda_\alpha)^{\beta-1} - (1 - \lambda_\alpha)^{\beta-1}}{2(\beta - 1)\lambda_\alpha(1 - \lambda_\alpha^2)^{\beta-1}} \right], \qquad \beta = \frac{1}{\alpha}.$$

(b) *The best starting approximation for m Newton iterations for the calculation of e^x is γr where r denotes the best relative approximation to e^x from $R_{m,n}[a, b]$ with deviation λ and*

$$\gamma = e \left(\frac{1 - \lambda}{1 + \lambda} \right)^{1/2\lambda} \frac{1}{(1 - \lambda^2)^{1/2}}.$$

The result (a) was obtained independently and simultaneously by D. L. Phillips [10] and G. D. Taylor [16]. Part (b) of the theorem is established in [6].

The aforementioned Property J is a rather natural analog of Property I for the case of uniform approximation. The Newton operators used for approximating $f(x) = e^x$ and $f(x) = x^\alpha$, $\alpha \in (0, 1)$, fail to possess Property J and hence Theorem 4.5 does not apply. However the Newton operator for approximating $f(x) = \ln x$ does possess Property J as is easily verified.

For $0 < a < b$, set

$$K = \{h \in C[a, b]: h(x) > \ln x - 1 \text{ for all } x \in [a, b]\}$$

and define

$$N_{\ln x}(h)(x) = h(x) - 1 + xe^{-h(x)}$$

for $h \in K$. It is straightforward to show that $N_{\ln x} : K \to K$, $N_{\ln x}$ is pointwise strictly monotone at $\ln x$ and is one-sided from above. With the above the following result can be established using the same techniques as in the proof of Theorem 5.1.

THEOREM 5.2. *Let $N_{\ln x}$ denote the Newton operator for approximating* $f(x) = \ln x$, *K as above and* $M = K \cap R_{m,n}[a, b]$ *where* $m \leqslant n$. *The best starting approximation (from M) for k Newton iterations for the calculation of* $\ln x$ *(in the uniform norm) is* $r^* + c_0$ *where* r^* *is the best uniform approximation to* $\ln x$ *from* $R_{m,n}[a, b]$ *with deviation* λ *(we assume* $0 < \lambda < 1$*) and* $c_0 = \ln[\sinh \lambda/\lambda]$.

6. FURTHER EXAMPLES OF OPERATORS DEFINING ITERATIVE SCHEMES

In the previous section we were concerned exclusively with iteration using Newton's method. We shall now consider other iterative methods for the solution of equations. First we introduce the important concept of order which affords us a means of classifying the iterative schemes which we shall discuss. The following definition is due to Traub [17]:

DEFINITION 6.1. An iteration function $\varnothing_f : E^1 \to E^1$ defined by $\varnothing_f(x_k) = x_{k+1}$, $k = 0, 1, 2,...$ for finding a root α of the equation $f^{-1}(y) - x = 0$ (fixed x) is said to be of order p if there exists a nonzero constant C such that $| \varnothing_f(x_k) - \alpha |/| x_k - \alpha |^p \to C$ as $k \to \infty$. The number C is called the asymptotic error constant.

It is clear that if the order of an iteration function exists, then it is unique. The iteration functions with which we are concerned are continuous one-point iteration functions; that is, each successive iterate x_{k+1} is obtained using only information from the previous iterate x_k. With each iteration function $\varnothing_f(x_k) = x_{k+1}$ we can, in a natural way, associate an iteration operator $\Phi_f : S \to S \subseteq C[a, b]$ via $\Phi_f(s_k) = s_{k+1}$ for approximating the function $f \in C[a, b]$. The subset S of $C[a, b]$ depends upon f and $[a, b]$. We shall now consider a number of iteration operators associated with known one-point iteration functions $\varnothing_f$ for $f(x) = e^x$ and $f(x) = x^{1/p}$, $p = 2, 3,...$. The one-point iteration functions $\varnothing_f$ which we shall be concerned with are order-preserving Padé rational approximations to a certain class of iteration functions generated by inverse hyperosculatory interpolation at a single point. A more detailed discussion of the above may be found in Traub [17].

We shall now enumerate various iteration functions with orders 2, 3, and 4. For each function we shall define the associated iteration operator and a suitable choice for the convex set K. In most instances K is chosen so that iteration is well-defined. We shall only remark that the operators $\Phi_f(f(x) = x^{1/p},\ p = 2, 3,...$ or $f(x) = e^x)$ considered are continuous, pointwise strictly monotone at f, possess Property I at f and have f as a fixed point. With the above information Theorems 4.3 and 4.6 allow us to calculate optimal starting approximations although no such calculations will be presented. Recall that if the operator is not one-sided then the optimal starting approximation may not be independent of the number of iterations.

Iteration Functions of Order 2

1. First we shall only mention that the Newton iteration function already discussed has order of convergence 2.

2. Iteration Function: $x_{n+1} = \varnothing_f(x_n) = (x_n/p)[(p+1) - (x_n{}^p/x)]$ for approximating $f(x) = x^{1/p}$, $p = 2, 3, \ldots$.

Convex Set: $K = \{h \in C[a, b]: 0 < h(x) < ((p+1)\, x)^{1/p}$ for all $x \in [a, b], 0 < a < b\}$.

Iteration Operator: $\Phi_f(h)(x) = (h(x)/p)\{(p+1) - [(h(x))^p/x]\}$, $h \in K$.

Properties: $\Phi_f : K \to K$, Φ_f is one-sided from below at $x^{1/p}$, $p = 2, 3, \ldots$.

Remark. If r_p denotes the best relative approximation to $x^{1/p}$ from $R_{m,n}[a, b]$ with deviation λ_p then we must require that

$$\lambda_p \leqslant [(p+1)^{1/p} - 1]/[(p+1)^{1/p} + 1]$$

in order that $\delta r_p \in M$ for $\delta \in [1/(1+\lambda_p), 1/(1-\lambda_p)]$.

Iteration Functions of Order 3

1. (a) For approximation of $f(x) = e^x$ on an interval $[a, b]$.

Iteration function:

$$x_{n+1} = \varnothing_f(x_n) = x_n[(2 + x - \ln x_n)/(2 - x + \ln x_n)].$$

Convex Set: $K = \{h \in C[a, b]: h(x) > e^{x-2}$ for all $x \in [a, b]\}$.

Iteration Operator:

$$\Phi_f(h)(x) = h(x)[(2 + x - \ln h(x))/(2 - x + \ln h(x))], \qquad h \in K.$$

Properties: $\Phi_f : K \to C[a, b]$ and so K needs to be further restricted to permit iteration.

Remarks. If r denotes the best relative approximation to e^x from $R_{m,n}[a, b]$ with deviation λ then we must require that $\lambda < (e^2 - 1)/(e^2 + 1)$ (which is a very mild restriction since we always have $\lambda < 1$) in order that $\delta r \in M$ for $\delta \in [1/(1+\lambda), 1/(1-\lambda)]$.

(b) For approximation of $f(x) = x^{1/p}$ on $[a, b]$, $0 < a < b$.

Iteration Function:

$$x_{n+1} = \varnothing_f(x_n) = x_n\{[(p-1)x_n{}^p + (p+1)x]/[(p+1)x_n{}^p + (p-1)x]\}.$$

Convex Set: $K = \{h \in C[a, b]: h(x) > 0 \text{ for all } x \in [a, b]\}$.

Iteration Operator:

$$\Phi_f(h)(x) = h(x)\{[(p-1)(h(x))^p + (p+1)x]/[(p+1)(h(x))^p + (p-1)x]\}, \quad h \in K.$$

Property: $\Phi_f : K \to K$.

2. (a) For approximation of $f(x) = e^x$ on $[a, b]$.

Iteration Function: $x_{n+1} = \varnothing_f(x_n) = \frac{1}{2}x_n[1 + (\ln x_n - x - 1)^2]$.

Convex Set: $K = \{h \in C[a, b]: h(x) > 0 \text{ for all } x \in [a, b]\}$.

Iteration Operator: $\Phi_f(h)(x) = \frac{1}{2}h(x)[1 + (\ln h(x) - x - 1)^2]$, $h \in K$.

Property: $\Phi_f : K \to K$.

(b) For approximation of $f(x) = x^{1/p}$ on $[a, b]$, $0 < a < b$.

Iteration Function:

$$x_{n+1} = \varnothing_f(x_n) = \frac{x_n(1-p)}{2p^2}\left[\left(\frac{2p-1}{p-1} - \frac{x}{x_n{}^p}\right)^2 + \frac{p^2(1-2p)}{(1-p)^2}\right].$$

Convex Set: $K = \{h \in C[a, b]: h(x) > 0 \text{ for all } x \in [a, b]\}$.

Iteration Operator:

$$\Phi_f(h)(x) = \frac{1-p}{2p^2}\, h(x)\left[\left(\frac{2p-1}{p-1} - \frac{x}{(h(x))^p}\right)^2 + \frac{p^2(1-2p)}{(1-p)^2}\right], \qquad h \in K.$$

Properties: $\Phi_f : K \to C[a, b]$ and so K needs to be further restricted.

Iteration Functions of Order 4

1. (a) For approximation of $f(x) = e^x$ on $[a, b]$

Iteration Function:

$$x_{n+1} = \varnothing_f(x_n) = x_n[1 - (\ln x_n - x) + \tfrac{1}{2}(\ln x_n - x)^2 - \tfrac{1}{6}(\ln x_n - x)^3].$$

Convex Set: $K = \{h \in C[a, b]: 0 < h(x) < e^{x+r_0} \text{ for all } x \in [a, b]\}$ where r_0 is the unique real zero of $1 - u + \frac{1}{2}u^2 - \frac{1}{6}u^3 = 0$.

Iteration Operator:

$$\Phi_f(h)(x) = h(x)[1 - (\ln h(x) - x) + \tfrac{1}{2}(\ln h(x) - x)^2 - \tfrac{1}{6}(\ln h(x) - x)^3]$$

for $h \in K$.

Properties: $\Phi_f : K \to K$, Φ is one-sided from below at e^x.

Remark. If r denotes the best relative approximation to e^x from $R_{m,n}[a, b]$ with deviation λ we must require that $\lambda < (e^{r_0} - 1)/(e^{r_0} + 1)$.

(b) For the approximation of $x^{1/p}$ on $[a, b]$, $0 < a < b$.

Iteration Function:

$$x_{n+1} = \varnothing_f(x_n) = -\frac{x_n}{p}\left[\left(\frac{x_n^p - x}{x_n^p}\right) + \frac{p-1}{2p}\left(\frac{x_n^p - x}{x_n^p}\right)^2 + \frac{(p-1)(2p-1)}{6p^2}\left(\frac{x_n^p - x}{x_n^p}\right)^3 - p\right].$$

Convex Set: $K = \{h \in C[a, b]: h(x) > 0 \text{ for all } x \in [a, b]\}$.

Iteration Operator:

$$\Phi_f(h)(x) = -\frac{h(x)}{p}\left[\left(\frac{(h(x))^p - x}{(h(x))^p}\right) + \frac{p-1}{2p}\left(\frac{(h(x))^p - x}{(h(x))^p}\right)^2 + \frac{(p-1)(2p-1)}{6p^2}\left(\frac{(h(x))^p - x}{(h(x))^p}\right)^3 - p\right]$$

for $h \in K$.

Properties: $\Phi_f : K \to K$, Φ_f is one-sided from above at $x^{1/p}$.

For the next two iteration functions we shall only be concerned with the approximation of e^x.

2. Iteration Function:

$$x_{n+1} = \varnothing_f(x_n) = x_n\left[1 - \frac{(\ln x_n - x)}{1 + \frac{1}{2}(\ln x_n - x) + \frac{1}{12}(\ln x_n - x)^2}\right].$$

Convex Set: $K = \{h \in C[a, b]: h(x) > 0 \text{ for all } x \in [a, b]\}$.

Iteration Operator:

$$\Phi_f(h)(x) = h(x)\left[1 - \frac{(\ln h(x) - x)}{1 + \frac{1}{2}(\ln h(x) - x) + \frac{1}{12}(\ln h(x) - x)^2}\right], \qquad h \in K.$$

Properties: $\Phi_f : K \to K$.

3. Iteration Function:

$$x_{n+1} = \varnothing_f(x_n) = x_n \left[1 + \frac{(\ln x_n - x)(\ln x_n - x - 6)}{2(\ln x_n - x + 3)}\right].$$

Convex Set: $K = \{h \in C[a, b]: h(x) > e^{x-3} \text{ for all } x \in [a, b]\}$.

Iteration Operator:

$$\Phi_f(h)(x) = h(x) \left[1 + \frac{(\ln h(x) - x)(\ln h(x) - x - 6)}{2(\ln h(x) - x + 3)}\right], \qquad h \in K.$$

Properties: $\Phi_f : K \to K$, Φ_f is one-sided from above at e^x.

Remark. We must require that $\lambda < (e^3 - 1)/(e^3 + 1)$, where λ denotes the error associated with r, the best relative approximation to e^x on $[a, b]$ from $R_{m,n}[a, b]$. As noted before, this restriction is extremely mild since we always have $\lambda < 1$.

Finally we shall examine an iteration function due to Kiss (see [17]) which also has order of convergence 4. As above we shall only consider the iteration function for approximating e^x on $[a, b]$.

4. Iteration Function:

$$x_{n+1} = \varnothing_f(x_n) = x_n \left[1 - \frac{(\ln x_n - x)(1 + \frac{1}{2}(\ln x_n - x))}{1 + (\ln x_n - x) + \frac{1}{3}(\ln x_n - x)^2}\right].$$

Convex Set: $K = \{h \in C[a, b]: e^{x-2} < h(x) < e^{x+2} \text{ for all } x \in [a, b]\}$.

Iteration Operator:

$$\Phi_f(h)(x) = h(x) \left[1 - \frac{(\ln h(x) - x)(1 + \frac{1}{2}(\ln h(x) - x))}{1 + (\ln h(x) - x) + \frac{1}{3}(\ln h(x) - x)^2}\right], \qquad h \in K.$$

Properties: $\Phi_f : K \to K$, Φ_f is one-sided from below at e^x.

Remark. We must require that $\lambda < (e^2 - 1)/(e^2 + 1)$, where λ is the error associated with r, the best relative approximation to e^x from $R_{m,n}[a, b]$.

REFERENCES

1. E. W. CHENEY, "Introduction to Approximation Theory," McGraw-Hill, New York, 1966.
2. R. F. KING AND D. L. PHILLIPS, The logarithmic error and Newton's method for the square root, *Comm. ACM* **12** (1969), 87–88.
3. H. L. LOEB, Un Contre-exemple a un Résultat de M. Claude Gilormini, *C. R. Acad. Sci. Paris, Sér. A* **266** (1968), 237–238.

4. H. L. LOEB, D. G. MOURSUND, L. L. SCHUMAKER AND G. D. TAYLOR, Uniform generalized weight function polynomial approximation with interpolation, *SIAM J. Numer. Anal.* **6** (1969), 284–293.
5. H. L. LOEB, D. G. MOURSUND AND G. D. TAYLOR, Uniform rational weighted approximations having restricted ranges, *J. Approximation Theory* **1** (1968), 401–411.
6. G. MEINARDUS AND G. D. TAYLOR, Optimal starting approximations for iterative schemes, to appear.
7. D. G. MOURSUND, Optimal starting values for the Newton–Raphson calculation of $\sqrt{x}$, *Comm. ACM* **10** (1967), 430–432.
8. D. G. MOURSUND AND G. D. TAYLOR, Optimal starting values for the Newton–Raphson calculation of inverses of certain functions, *SIAM J. Numer. Anal.* **5** (1968), 138–150.
9. A. PERRIE, Uniform rational approximation with osculatory interpolation, Doctoral Thesis, University of Oregon, 1969.
10. D. L. PHILLIPS, Generalized logarithmic error and Newton's method for the *m*th root, *Math. Comp.* **24** (1970), 383–389.
11. J. R. RICE, "The Approximation of Functions," Vol. I, Addison–Wesley Co., Reading, MA, 1964.
12. L. SCHUMAKER AND G. D. TAYLOR, On approximation by polynomials having restricted ranges, MRC Tech. Summary Report 835, Madison, WI, January, 1968.
13. P. H. STERBENZ AND C. T. FIKE, Optimal starting approximations for Newton's method, *Math. Comp.* **23** (1969), 313–318.
14. G. D. TAYLOR, Approximation by functions having restricted ranges III, *J. Math. Anal. Appl.* **27** (1969), 241–248.
15. G. D. TAYLOR, On approximation by polynomials having restricted ranges, *SIAM J. Numer. Anal.* **5** (1968), 258–268.
16. G. D. TAYLOR, Optimal starting approximations for Newton's method, *J. Approximation Theory* **3** (1970), 156–163.
17. J. F. TRAUB, "Iterative Methods for the Solution of Equations," Prentice–Hall, Inc., Englewood Cliffs, NJ, 1964.